Section 4.3: Riemann Sums and Definite Integrals

Riemann Sums

The definition of a Riemann sum is the same as that of the area formula we used in section 4.2 with the following generalizations:

- The bases of the rectangles no longer need to be of equal. Thus Δx becomes Δx_i. This makes no difference to the sun as long as the *largest* base (called the *norm* and denoted $||\Delta||$) approaches zero.
- The function no longer needs to be nonnegative; now functions can have negative sums.
- The function no longer needs to be continuous; we insist only that it is defined over the interval in question.

$$\sum_{i=1}^{\infty} f(c_i)\Delta x_i$$

This broader definition allows summation of some functions that couldn't be done otherwise (see ex. 1, p. 271).

Definite Integrals

[The definite integral is defined as the limit of the Riemann sum of a function over a closed interval [*a*,*b*] as the norm approaches zero.]

$$\lim_{||\Delta|| \to 0} \sum_{i=0}^{\infty} f(c_i)\Delta x_i$$

This is denoted by

$$\int_a^b f(x)dx$$

where a and *b* are referred to as the lower and upper limits respectively.

Despite the similarity of notation, the indefinite integral (antiderivative) is a family of functions, while the definite integral is a number. We will see how these two concepts are related in the next section.

Although the formal definition of the integral is rather complicated, many definite integrals can be calculated using shortcuts like the ones we used in section 4.1. However, we will not need them for this section.

All continuous functions are integrable, but not all integrable functions are continuous.

Although the definite integral is not an area according to the definition, it can still be used to calculate areas. To calculate the area of a region below the *x*-axis, simply take the integral and multiply it by -1.

Properties of Definite Integrals

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b k f(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

`[Homework: p. 278 #1, 3, 9, 15, 25, 29, 41, 42, 45, 55, 56, 69, 70]`

Section 4.4: The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus states that the definite integral is equal to the *change* in the indefinite integral of a function over a certain interval.

In other words, the definite integral of a function is equal to the difference between an antiderivative when evaluated at the upper and lower limits.

$$\int_a^b f(x)dx = F(b) - F(a)$$

This can be proven by using the Mean Value Theorem (see p. 283).

Roughly speaking, the FTOC means that the definite integral is the "inverse" of differentiation. This result was anticipated when we used similar notation to write the definite and indefinite integrals, even though we arrived at their definitions by very different routes.

Note: The right hand side of the previous equation is often abbreviated as follows:

$$F(x)\big]_a^b$$

See page 284 for examples of how to evaluate definite integrals.

The Mean Value Theorem for Integrals

For any function $f(x)$ defined over the interval $[a,b]$, there exists a point c such that $f(c)$ is equal to the average value of the function of over that interval.

$$f(c)=\frac{1}{b-a}\int_a^b f(x)dx$$

If we apply the FTOC, we see that this is equivalent to the Mean Value Theorem (p. 174) except that now the left side is a function rather than a derivative of a function.

$$f(c)=\frac{F(b)-F(a)}{b-a}$$

The Mean Value Theorem for Integrals may be interpreted as follows: For any function defined over a particular interval, there exists a rectangle whose base is the interval and whose area is equal to the definite integral over that interval.

The Second Fundamental Theorem of Calculus

If the limits of a definite integral contains a variable, then the integral is a function of that variable. (Note that the variable of integration exists only in the integrand and therefore is a dummy variable.)

A function of the form

$$F(x)=\int_a^x f(t)dt$$

is called an accumulation function because it can be interpreted as the area accumulated in the interval from *a* to *x*.

The Second FTOC states that taking the derivative of an accumulation function yields the original function.

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right]=f(x)$$

Here, the value of *a* is irrelevant because taking the derivative will cancel out the terms it creates.

The Second FTOC is useful because it can be used to find the derivatives of integral functions even if the integral itself cannot be evaluated using conventional techniques (e.g. #91, p. 293)

[Homework: p. 291 #

Section 4.5: Integration by Substitution

The chain rule states that the derivative of a composite function *F*(*g*(*x*) is $F'(g(x))g'(x)$.

In order to integrate some functions, one can manipulate the integrand so that it is of the form *F*'(*g*(*x*))*g*'(*x*). Then, according to the chain rule, its integral is

$$\int F(g(x))g'(x)dx=F(g(x))+C$$

This process can often be simplified by replacing *x* in the previous expression with a different variable *u*=*g*(*x*) and integrating in terms of *u*. [See, for example Example 5, p. 298]

Identifying *u*=*g*(*x*) is critical, because all the *x*'s in the original expression must replaced by *u*'s. Keep an eye out for compound functions and common derivatives in the integrand. These will reveal *u*'s identity.

Method for integration using change of variables

1. Choose a substitution $u = g(x)$.
2. Compute du = $g'(x)$dx
3. Rewrite the integral in terms of *u* and d*u*.
4. Integrate in terms of *u*
5. Replace *u* with *g*(*x*).
6. If the integral is definite, evaluate at the limits

When computing a definite integral, it is often simpler to skip step 5 and evaluate the derivative in terms of *u*=*g*(*x*) instead of *x*. When this is done, the limits of integration must be changed to *g*(*b*) and *g*(*a*) instead of *b* and *a*.

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

<u>Integration of Even and Odd Functions</u>

If f is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

If f is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

[Homework: p.304: #...

Section 4.6: Numerical Integration

Some functions have integrals which cannot be written in terms of elementary functions. In these cases, the definite integral may still be approximated by using Riemann sums or by interpreting the integral as an area and using the methods of section 4.2. (This is the way most graphing calculators integrate functions.) However, when integrating by hand these approximations may be be inaccurate unless region is partitioned into a large number rectangles.

Fortunately, there are methods for calculating integrals numerically which require fewer terms.

- The T*rapezoidal Rule* treats each term in the sum as a trapezoid rather than a rectangle. As before, the sum of the areas of the trapezoids (including negative "areas") is approximately equal to the integral. Its formula derived on p. 309) is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + +f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

The terms have coefficients according to the pattern 1   2   2   2 ... 2 1

- *Simpson's Rule* treats each term in the sum as the region under a parabola.

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}[f(x_0)+\ 4f(x_1)+ +2f(x_2)+4f(x_1)+\cdots+4f(x_{n-1})+f(x_n)]$$

Here, the coefficients accord to the pattern 1   4   2   4   2   ...   2   4   1

<u>Error Analysis</u>

For a continuous function, the error (difference between the integral and the approximation) can be made arbitrarily small by increasing the number of terms (Theorem 4.19)

The maximum error when using the trapezoidal rule is inversely proportional to the square of the number of terms.

$$E \le \frac{(b-a)^3}{12n^2} \cdot max|f|$$

The maximum error when using Simpson's rule is inversely proportional to the fourth power of the number of terms.

$$E \le \frac{(b-a)^5}{180n^4} \cdot max|f^{(4)}(x)|$$

[Homework: p. 314: #...